**Renormalization Group**

**Some Definitions**

Let’s have in mind the discrete Ising model action,



where,



or the continuous Ising action,



where (z = # nearest neighbors),



or continuous Heisenberg O(n) action,



where (κ = n, z = # nearest neighbors),



or NLSM,



where,



Recall J is the coupling constant between spins, h is the ‘magnetic field’ (we shoved some extra terms in there to give it units of energy – see Ising Weiss homogeneous). And we’re using units where kB = 1 obviously.

**Standard Mean Field and Perturbation Theory Breakdown**

So when we applied MFT to our spin problem, we found predictions for a phase transition in all dimensions, and α, β, γ, δ, in particular, all had the same values in all dimensions. Turns out this is correct for d ≥ 4 (the upper critical dimension). But not only are the exponents wrong for d < 4 (though in the ballpark for d = 3), the phase transition doesn’t even happen for d < 2. When we incorporated fluctuations into our calculations, we saw there *was* a dimensional dependence to the critical exponents for d < 4, though not for d > 4 (well γ was the only exponent we really looked at), and that the phase transition was probably not going to happen for d < 2 as the critical temperature → 0 as d → 2, and some (all?) of the critical exponents, (again only really looked at γ), took on unphysical values for d < 2. All of this was a big step in the right direction. Problem so far is that the predictions for the critical exponents in d = 3 really aren’t better when incorporate fluctuations. And also, our perturbative calculation of the changes of critical exponents due to the uφ4 term was non-perturbative, i.e., it didn’t depend on u. And when we did the self-consistent HF approach, we also found the corrections didn’t depend on u. And were also not correct FWIW. This seemed to indicate two things. One, the uφ4 term isn’t small close to the transition, even though <φ> *is*, evincing that the fluctuations in φ are really large, and so <φ4> is really large. This is why peturbation theory breaks down. And even the SCHF approach didn’t fix this, because I guess we weren’t including the proper set of self-energy terms in our calculation. Note that even so, our MF solution (and its corrections) work well when they’re outside a *small* window about TC, where fluctuations *are* small. So options at this point would be to find some other appropriate way to sum up diagrams. I’m not sure what this would be per se. Or we can try…

**RG Approach**

RG seems to take an indirect approach to analyzing the system near the critical point. It takes advantage of the fact that the correlation length goes to infinity as we approach the critical point. And so the system kind of looks the same at the microscopic and macroscopic level. This self-similarity enables us to construct a scaling equation for the free energy, and its derivatives, etc. This will provide some justification for the Widom scaling laws discussed in the Thermodynamics/Critical Exponents file. This is very similar to how in the Thermodynamics folder/Thermodynamics Potentials file, we were able to use the homogeneity of the potentials to construct a scaling equation for them, and then infer properties of the potentials. So once we get this scaling equation, we can use it to put the free energy, etc., in the form of Widom’s scaling laws. And then by comparison we can extract the critical exponents. It is useful too in the sense of showing us which terms are necessary close to the critical point when going from a finite to continuous model. Perhaps such an analysis would show what exactly the shortcomings would be in the Anderson model continuum limit case.

**Coupling constant flows under rescaling**

Suppose we have a partition function Z for a lattice model with spacing *a*, and coupling constants **α**. And then say we make a partial sum over all variables φ(xa) within lattice block of length b*a*, subject to constraint that Σφ(xa) = φb.



Then we’ll have a partition function on a lattice model with spacing b*a*, which tells us how these new spin *clumps* interact with each other.The crucial assumption is that this will result in an H of the same form, but with renormalized coupling constants **α**(b). For concreteness, let’s say we’re dealing with something like the Ising or Heisenberg models,



where (κ = n, z = # nearest neighbors),



Our starting values we’ll call r = r1, j = j1, u = u1. These are the ‘physical’ values of our parameters, for the system with the actual/physical lattice spacing *a*. And when we sum over the cell with length ba, we’ll get renormalized coupling constants rb, jb, ub.

But some times, we will find that the same exact H form doesn’t return and more/different ‘interactions’ emerge, like some vφ6 tterms, for instance. If such a term doesn’t scale away to irrelevance as b → ∞, meaning vb → 0, then we have to start over with said vφ6 term in our bare H and see how everything renormalizes again. Hopefully *this* time we’ll get the same Hamiltonian as the bare one, but with renormalized coupling constants vb. But if an extra term arises *again*, say, wφ8, and doesn’t scale away, then we have to include *that* term in the bare Hamiltonian, and repeat the processes, getting some renormalized wbφ8. And so on, until that doesn’t happen anymore. But anyway, say we’ve identified r, j, and u as coefficients of the ony relevant terms. Then our goal is to find an equation for these renormalized coupling constants,



where r1, j1, and u1 are values of the constants when b = 1. A little digression…note that these equations will be ‘length-independent’ in a sense. There is nothing *mathematically* special *per se*´ about the building block size, L, at the starting point. Typically it would be L = a. And in this case the initial values r1, j1, u1 would be the actual physical parameter values of our system: r1 = n-Jz/T where T is the actual temperature , j1 = βh where h is the actual field (or actual 0.5gγB I think), and u1 = uis the actual whatever coupling constant. But the scaling relationship will be general, like the β(g) function thing in the Anderson localization stuff. So say our starting building block size is L = na (where n is some integer), then in terms of the renormalized building block *length* bL, our equations would read:



where f(r,j,u) are all still the same functions and r1(L), j1(L), u1(L) would be the renormalized parameter values at the length scale L = na … end digression. We usually don’t start off knowing the formula for the renormalized coupling constants. Typically we’ll find a coupled set of differential equations for coupling constants, of the following form:



where the beta functions βr,j,u will presumably be known. Together with specified initial conditions r1, j1, u1, we can solve these equations in principle. And so we could then see the path (rb, jb, ub) traces out in their ‘phase’ space.

What do we expect? Generically, as we increase b, we are ‘zooming’ out from the system. And in that case, the fluctuations will seem less severe, since we’ll be averaging over larger effective cell size. Fluctuations are greatest near the critical point. So one expects that as fluctuations decrease, we’re leaving the critical point. So under renormalization, our coupling constants should scale away from the critical point. So if we start at a temperature above the critical point, then we’d expect renormalization to result in further increasing temperature since this means less fluctuations because the spins more uniformly point randomly; there is less correlation between neighboring spins. If we start below the critical temperature, then we’d expect the temperature to further decrease, as this also means less fluctuations because the spins get frozen into their ordered state. And in either case, if we start with non-zero field, we’d expect the field strength to get higher under renormalization (either in positive or negative direction), as this also would be congruent with decreasing fluctuations. Since r and j increase w/o bound in the positive or negative directions, they’re called relevant parameters. What about the coupling constant u? To be consistent with Widom’s scaling hypothesis, we’ll see that u must go to a constant under renormalization. And it’s called therefore an irrelevant parameter. Parenthetically, if the constant it goes to is 0, then it’s called a dangerously irrelevant parameter. *Dangerous* because thermodynamic quantities often depend on u in such a fashion that if u → 0, we’d need delicate cancelations to avoid having f, m, etc., → ∞. And if u were to not change at all, then it’d be called a marginal parameter. Now it is possible that under renormalization, r, j, and u do not change at all. Values (r\*, j\*, u\*) where this happens are called *fixed* points. Such a point would be characterized by:



The fixed point would seem to correspond to the critical point, since when we’re at the critical point, ξ diverges, and the system becomes self-similar at all length scales, so that we wouldn’t imagine the coupling constants would change upon rescaling. And in fact it is the critical point, rc = r\*, jc = j\*, *for* the parameter value u = u\*. In general, there will separate critical points for every separate value of u. This set of critical points is called the critical hypersurface (or line). These can be identified as the points, which, under rescaling, flow *directly* into the fixed point (r\*, j\*, u\*). This is because if these points flow into the fixed point, then their correlation length, ξ, must also be ∞, since ξ always decreases under rescaling, and yet is equal to ∞ at the fixed point, as we said above. Of course if there is no extra parameter, u, in our model, then r\* and j\* will just be *the* critical point. We’ll be especially concerned about the behavior of the coupling constants near this fixed point. We’ll find it sufficient (so far) to linearize the equations near the fixed point (subscript stands for derivative). Taylor expanding our RHS of the equations gives us:



which we can put in matrix form. For short, defining |q> = (rb, jb, ub), |q\*> = (r\*, j\*, u\*), and |Δq> = |q> - |q\*>, we can say,



and is the matrix of β coefficients above. usually won’t be symmetric, and so we wouldn’t seem to be guaranteed eigenvectors and eigenvalues, but it appears that it usually will have a complete set of eigenvalues and vectors. Let these be λ and |λ>. Then we can formally write the solution (see Appendix)



So,



From Widom’s scaling hypothesis, we’ll see that we expect two of the eigenvalues to be positive, specifically the ones governing the evolution of the temperature (or 1/Temperature), λr, and field, λj, i.e., in this example, of rb and jb. Doesn’t seem like there is a requirement that the eigenvectors, |λr> and |λj>, corresponding to these two eigenvalues should strictly point along the r and j axes, as they are depicted to. Seems like they could point anywhere in the r-j plane. But either way, since their eigenalues are positive, they *will* direct flows away from the critical point – hence the arrows. And we’ll expect the eigenvalue, λu, governing the evolution of the extra coupling constant ub to be negative. The eigenvector corresponding to ub, |λu>, should then point along the direction of critical hypersurface (line). This is because if a point lies on this line, then its evolution will be given by |qb> = |q\*> + |λu><λu|Δq1>bλ\_u, and so, since λu < 0, it will flow into the fixed point |q\*>. Hence the arrows are pointed inward towards the fixed point.

Diagram

Description automatically generated

Next question is how we get the scaling equation for the free energy, and then how we extract the critical exponents from it.

**f, ξ, m, etc. scaling equations and critical exponents**

So consider a model of some system, with lattice spacing *a*. We can calculate its free energy density. Let F(t,j,u) be its free energy per unit cell. Then we could say the free energy *density* (times β) would be f(t,j) = β(t)F(t,j,u)/ad (and we’ll synonymously refer to t,j,u as t1,j1,u1). Actually, I’ll just say f(t,j) = βcF(t,j,u)/ad, since we’ll be interested in the T → Tc limit. And then we can do the partition function sum over volume (b*a*)d, and we should get some new free energy formula in terms of renormalized coupling constants. Now the free energy over the same unit cell volume ad would be f(t,j) = βcF(tb,jb,ub)/(b*a*)d. Note u is not a ‘thermodynamic variable’, just a parameter. So we should have:



where,



if we’re near the critical point. This is our scaling equation for the free energy, basically. Now fill in the expected form of the solution. Well what is this? Consider a purple vector |Δq1> = |Δr1, Δj1, Δu1> that lies close to the critical hypersurface. And note that the critical temperature and field would be given by the point (kind of barely discernable) on the critical hypersurface corresponding to the point u1.

Diagram

Description automatically generated

The blue ray gives the path of this point under rescaling. It’s given by the equation,



I attempted to depict the <λ|Δq1> overlaps in pink. The diagonal line, along the critical hypersurface is <λu|Δq1>, and the vertical pink line along the r direction is <λr|Δq1>, and there would also be an overlap <λj|Δq1>, but it’s not pictured really.

A diagram of a graph

Description automatically generated

So <λu|Δq1> is very nearly given by the magnitude of |Δq1>, i.e., √[(Δr1)2+(Δj1)2+(Δu1)2]] ≡ c. The other overlaps are nearly zero, and they go to zero as a function of (r1-rc) and (j1-jc). So we may write:



If we expand for small r1-rc and j1-jc, we’d get something like,



Now come some assumptions. First, even though I’ve preemptively associated one of the positive eigenvalues/eigenvectors with λr/|λr>, and the other with λj/|λj>, I’m not sure it would be clear which goes with which to start with. If these two eigenvectors just pointed in random directions in the r-j plane, how would we know which to associate with ‘r’, and which to associate with ‘j’? Well I’m going to assume that these eigenvectors will turn out to point along the r and j axes respectively. Then |λr> would be (1 0 0) and |λj> = (0 1 0). And further I’ll presume that c1j = 0, and c2r = 0. Might not need to make these assumptions, but they often seem to be made…then we can write:



Finally, let’s fill these into our free energy, remembering r1, j1, u1 are just our actual/physical r, j, u:



now,



and presuming jc = 0, as it will be in all our examples, we can say,



And I guess I’ll absorb the -c1rKc, and c2j into F to write:



Now compare to Widom’s scaling law close to the critical point (see Thermodynamics / Critical Exponents file),



We can match this form with the choice, b = 1/|t|1/λ\_r (and note that close to the critical point, where t = 0, b will be large, so ex post facto, that’s why we’ve been taking the large b limit). So the RG group method can be said to derive Widom’s scaling laws (the fact that the exponents in his law are 2-α and Δ can be ascertained merely by calculating, say, the heat capacity and such; then we’d see, e.g., that it must be *α* in 2-α, as shown in that Thermodynamics/Critical Exponents file). So now we have,



And so we conclude:



From these, all the other exponents come from critical exponent relations in the Thermodynamics folder/Critical Exponents file. But for the sake of discussion, let’s explicitly consider the order parameter – magnetization say. What’s the coupling constant scaling equation for the magnetization? Well from above, it’s,



where we make a definition of the function M in the last line. And let’s compare to the magnetization in zero field,



we can match forms if we say, again, that b = 1/|t|1/λ\_r. So we have:



So we have:



Another route to the same thing is to consider the correlation length. We can say the correlation length per unit cell is given by some function X(r,j,u). Then, making the same manipulations we did on f,



Now consider the form of the scaling relation (see Thermodynamics/Phase Transitions & Critical Exponents file, again):



So we say, let b = |t|-1/λ\_r once more. Then we have:



So again we see,



And all the other critical exponents follow from (see Thermodynamics/Critical Exponents file):



and if we’re in d < dupper\_critical, which is four for the Ferromagnetic transition, then we can add the hyperscaling equation:



And combining the 2 – α = d/λr equation with the ν = 1/λr equation results in the hyperscaling equation. But can see that it does so regardless of dimension. This is one indication of a potential problem in our analysis relating the critical exponents to the eigenvalues if d > dupper\_critical.

**Dangerously irrelevant operators**

So in the abstract examples above, we presumed the scaling of the supposedly irrelevant variable went as u(b) = u\* + (u – u\*)bλ\_u. But what if u\* = 0? This is when u is called a dangerously irrelevant variable I think, and I this often (always?) happens for d > duc. And then we have to be more careful with our analysis. This will change our result for the equation relating α to the eigenvalues, and consequently all the eigenvalues based on that equation. But it won’t change the equation relating ν to the eigenvalues. And as we’ll see this change will result in the hyperscaling relation failing, as it’s supposed to, in d > duc. So for instance, say we’re looking at the free energy,



And now like before we’ll just absorb the -c1rKc, and c2j into F, and let’s do same with c3:



Now compare to Widom’s scaling law close to the critical point (see Thermodynamics/Critical Exponents file),



We can match this form with the choice, b = 1/|t|1/λ\_r (and note that close to the critical point, where t = 0, b will be large, so that’s why we’ve been taking the large b limit). So now we have,



Now it will often happen that limx→0F(a,b,x) will not go to a constant F(a,b,0). So the limit cannot be evaluated until we know the functional form. Usually, this would not be known a priori. But I guess these issues will usually arise for d > dupper\_critical where MFT prevails. And so we can work out the functional form using, well, MFT. Actually, the free energy is kind of hard to get in full generality near the critical point, so we’ll specialize to the case j = 0. As we saw in the Ising (HS Mean Field Inhomogeneous file), the free energy goes as F ~ r2/u. So,



Therefore,



And now let’s look at the order parameter. From our scaling analysis, we’d say (see above):



Comparing to,



we can match forms if we say that b = 1/|t|1/λ\_r. Then,



Now we have to evaluate the limit limx→0M(tc,0,x). Well in the Ising-HS MFT (inhomogeneous) file, we found for the magnetic moment of a unit cell (t and r are proportional near Tc, and I’m going to ignore the proportionality factor, and we know M is equal to this since when b = 1, we get m = M/ad),



That being the case, we can proceed with the limit,



So then we’d conclude:



Now consider the magnetization with t = 0. Our generic scaling equation is:



Comparing to,



we can match forms if we say that b = 1/j1/λ\_j. Then,



Now we have to evaluate the limit limx→0M(0,1,x). Well in the Ising-Weiss MFT (inhomogeneous) file, we found for the magnetic moment of a unit cell (t and r are proportional near Tc, and ignoring the proportionality factor),



That being the case, we can proceed with the limit,



So then we’d conclude:



Finally, let’s reconsider the correlation length. Recall we can say the correlation length per unit cell is given by some function X(K,j,u). Then,



where we once again make the presumption that u\* = 0. Now consider the form of the scaling relation in zero field:



So we say, let b = |t|-1/λ\_r again,



And now we must evaluate the limit. So I’ll presume MFT prevails in this regime where we get u\* = 0. And in that case, we can look to the Weiss-MFT-inhomogeneous file, where we found that ξ goes as:



So actually, the limit ubλ\_u­­­ → 0 is well defined, and we obtain just as before.



We can make a similar analysis for the other critical exponents. But we can now get all the rest from the general Widom scaling relations:



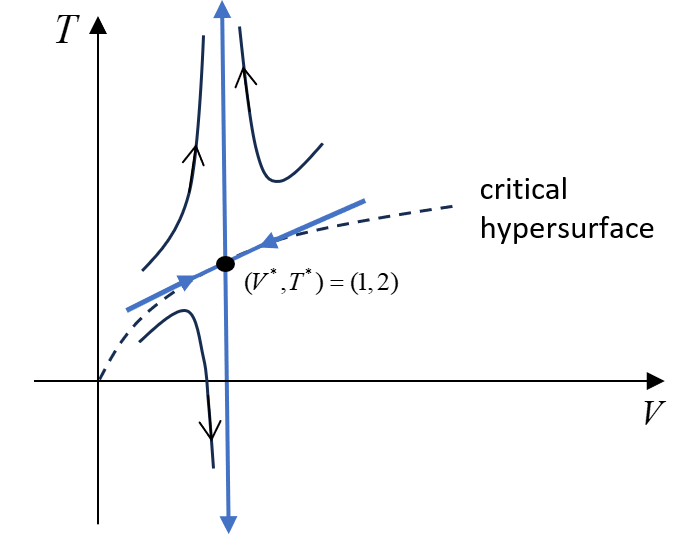
But we’ll note that this time the hyperscaling equation won’t work out.

**Summary**

So the renormalization group analysis shows the coupling constants scale with b. The scaling equations validate Widom’s scaling hypothesis for the thermodynamic potentials, and their derivatives. For d < duc, they consequently reproduce the critical exponent relations, as well as the hyperscaling equation. For d > duc, if we properly account for dangerously irrelevant variables, then we’ll see that the (dimension-independent) mean field critical exponents are reproduced, and the hyperscaling equation is invalidated.

**Example**

Consider the RG diagram below, where T = temperature and V is an interaction parameter, each in reduced units. The critical point is displayed, and you can assume the dotted curve is proportional to √V.



If we have a system prepared for which V = 5, what is the critical temperature? We need to figure out the formula for the hypersurface. Since it goes through the point (1,2), and we know that T(V) = A√V, we can say 2 = A√1 🡪 A = 2. So T(V) = 2√V. Now when V = 5, the critical temperature will be Tc = 2√5.

**Appendix**

Let’s work out the eigenvalue decomposition of a triangular matrix, A. So let:



Can see that eigenvalues are given by λ1,2 = a11, a22. Eigenvectors are?



So the eigenvector matrix is:



The eigenvectors are the columns of U, and their adjoints are the rows of U-1. So,



Obviously since A isn’t Hermitian, this looks a little different than we’re used to. But we’ll note that <λi|λj> = δij nonetheless. So can we write:



That checks out. And can we say,



Yes. And we can say?



So that checks out too. So looks like we can just copy all the stuff we know from regular Hilbert spaces. We just have to acknowledge that |λi> and <λi|are related a little differently than before. Now let’s consider the solution of the equation:



and is the matrix and |Δq> is a two component vector (qa, qb). usually won’t be symmetric, and so wouldn’t seem to be guaranteed to have eigenvectors and eigenvalues, but we already saw that if A is triangular then we do. And it seems that it usually will have a complete set of eigenvalues and vectors. Let these be |λ> and λ. So then we can formally write the solution,



And so we have a result familiar from QM,



And written out, explicitly, for each component,



which is:



Just to check that our solution is the solution,



So yyyeah. The result for Δq(b) isn’t surprising, and again, this could’ve been done pretty easily w/o the whole eigenvector decomposition thing.